

# Midi-Superspace Models of Canonical Quantum Gravity\*

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## Abstract

A midi-superspace model is a field theory obtained by symmetry reduction of a parent gravitational theory. Such models have proven useful for exploring the classical and quantum dynamics of the gravitational field. I present 3 recent classes of results pertinent to canonical quantization of vacuum general relativity in the context of midi-superspace models. (1) I give necessary and sufficient conditions such that a given symmetry reduction can be performed at the level of the Lagrangian or Hamiltonian. (2) I discuss the Hamiltonian formulation of models based upon cylindrical and toroidal symmetry. In particular, I explain how these models can be identified with parametrized field theories of wave maps, thus a natural strategy for canonical quantization is available. (3) The quantization of a parametrized field theory, such as the midi-superspace models considered in (2), requires construction of a quantum field theory on a fixed (flat) spacetime that allows for time evolution along arbitrary foliations of spacetime. I discuss some recent results on the possibility of finding such a quantum field theory.

## I. INTRODUCTION

A time-honored strategy for extracting information from a field theory is to restrict attention to states of the system that possess some degree of symmetry. This strategy was applied to canonical quantum gravity by DeWitt in 1967 [1]. He studied the canonical quantization of spacetimes with matter that were homogeneous and isotropic. This symmetry assumption turns an intractable problem in quantum field theory into a straightforward quantum mechanical problem that can be solved completely and illuminates some of the qualitative features (conceptual if not technical) of the full theory. A couple of years later, Misner dropped the isotropy assumption and studied classical and quantum dynamics of homogeneous spacetimes [2]. The field of quantum cosmology was born.

When using a description of gravitational dynamics based upon metrics, it is reasonable to try to represent the quantum states of the gravitational field as functions of three-dimensional spatial geometry [1]. This domain for the quantum gravitational state function has come to be known as “superspace”. Naturally, if one restricts attention to a small subspace of superspace, such as is done in quantum cosmology, one naturally is dealing with a “mini-superspace”. The mini-superspace models of gravity consider spacetimes with so much symmetry that there are only a finite number of degrees of freedom left in the gravitational field. A few years after DeWitt and Misner began the mini-superspace/quantum cosmology program, Kuchař went one step further [3]. He considered the quantization of spacetimes admitting cylindrical symmetry. This symmetry assumption was strong enough for him to make progress in understanding canonical quantum gravity, but weak enough so that the number of degrees of freedom in the gravitational field were still infinite. Indeed, the symmetry assumptions made in [3] reduce the gravitational dynamics to that of Einstein-Rosen waves. Borrowing from the vernacular of the fashion world, Kuchař proposed to call this model a “midi-superspace” model. Whether discussing mini-superspaces or midi-superspaces, one is referring to symmetry reductions of the gravitational field. Mini-superspaces lead to mechanical models; the reduced field equations become ordinary differ-

ential equations. Midi-superspaces define field theories; the reduced field equations remain partial differential equations.

Since the pioneering work of DeWitt, Misner, and Kuchař, the amount of effort spent in studying classical and quantum properties of symmetry reductions of general relativity (and other field theories) has been enormous. My purpose here is not to review this sizable body of literature but rather to highlight some recent developments in the area. Of course, the choice of material has a strong editorial bias. Most of the work that I will discuss was performed in collaboration with others. In particular I have benefited from working with Ian Anderson, Mark Fels, Joseph Romano, and Madhavan Varadarajan on the topics discussed below. Needless to say, I take credit for the inevitable weaknesses in the presentation.

The following 3 topics will be discussed here.

(1) *The Symmetric Criticality Principle*: What are necessary and sufficient conditions such that the canonical structure of a field theory induces a canonical structure on a given symmetry reduction of the theory? This issue is clearly relevant when one is studying canonical quantization of a mini- or midi-superspace, since one needs to know what is the Hamiltonian, what are the constraints, what are the Poisson algebras of various functions, *etc.* The issue boils down to the question: When can one symmetry reduce a classical field theory via a symmetry reduction of the Lagrangian? In the mathematics literature this question has appeared as the question of the validity of the “symmetric criticality principle” [4].

(2) *Two Killing Vector Midi-Superspaces*: I will consider some relatively recent results on the Hamiltonian structure of a popular class of midi-superspace models in which one assumes cylindrical symmetry or toroidal symmetry. These are examples of “2 Killing vector models”. The results presented are generalizations of the work of Kuchař, and lead to a nice scheme for canonical quantization.

(3) *Quantization on Curved Surfaces*: The models considered in (2) allow one to turn the problem of finding a gauge-invariant canonical quantization of the 2 Killing vector models into the problem of quantizing certain parametrized field theories. The problem of quantizing

parametrized field theories is an interesting problem in its own right. Dirac seems to be one of the first to study this problem [5]; he called it the problem of *quantization on curved surfaces*. I will present some salient results in this area.

## II. THE SYMMETRIC CRITICALITY PRINCIPLE

Symmetry reduction of a classical field theory takes place in 3 steps. First, one specifies a group action with respect to which the fields are to be invariant. Second, one constructs the most general field admitting the chosen group action as a symmetry. This is the *invariant field*. Normally, the invariant field involves arbitrary functions of one or more variables. These are the *reduced fields*, which define the mini- or midi-superspace. Third, one evaluates the field equations on the invariant fields thus obtaining the differential equations (or perhaps algebraic equations) for the reduced fields. These equations are the *reduced field equations* for the mini- or midi-superspace. All of us at one time or another have performed symmetry reduction in this way, *e.g.*, using time translation and rotational symmetry to find the Schwarzschild solution of the vacuum Einstein equations.

Normally, the field equations can be derived from a Lagrangian, in which case there is a very tempting shortcut one might try to obtain the reduced field equations. One can try substituting the invariant field into the Lagrangian, thereby obtaining a *reduced Lagrangian* for the reduced fields. One can then compute Euler-Lagrange equations from this reduced Lagrangian and obtain a set of reduced field equations. The only difficulty with this shortcut is that there is no guarantee the reduced Lagrangian will yield the correct reduced field equations! This problem is particularly vexing when studying quantization of mini- or midi-superspaces since one would like to assume that restriction of the Lagrangian (or Hamiltonian) to the invariant fields yields the correct Lagrangian (or Hamiltonian) for the mini- or midi-superspace.

A formal statement of the issue at hand is as follows. Let  $\mathcal{Q}$  be the space of metrics  $g$  on a manifold  $M$ . Let  $S: \mathcal{Q} \rightarrow \mathbf{R}$  be an action functional invariant under some transformation

group  $G: \mathcal{Q} \rightarrow \mathcal{Q}$ . Let  $\hat{\mathcal{Q}}$  be the space of  $G$ -invariant metrics  $\hat{g}$  and let  $\hat{S}: \hat{\mathcal{Q}} \rightarrow \hat{\mathcal{Q}}$  be the restriction of  $S$  to the invariant metrics. We want to know if the critical points of  $\hat{S}$  define critical points of  $S$ , that is,

$$\left. \frac{\delta S}{\delta g} \right|_{\hat{\mathcal{Q}}} = 0 \iff \frac{\delta \hat{S}}{\delta \hat{g}} = 0. \quad (2.1)$$

Put simply: we want to know if symmetric critical points are critical symmetric points. If this is the case, following Palais, we say that the *symmetric criticality principle* holds [4]. In essence, if the symmetric criticality principle holds then one can expect to describe the reduced field theory using the reduced Lagrangian. Let us look at a couple of examples.

First we consider the spherical symmetry reduction of vacuum general relativity. We restrict attention to spacetimes admitting a three dimensional isometry group  $G$  that is isomorphic to  $SO(3)$  and has orbits that are spacelike and diffeomorphic to two-dimensional spheres. It follows that there will exist (non-unique)  $G$ -invariant functions,  $t \in (-\infty, \infty)$  and  $r \in (0, \infty)$ , which can be used as coordinates and such that the spacetime metric  $g$  takes the form

$$g = A(r, t)dt \otimes dt + B(r, t)dt \otimes dr + C(r, t)dr \otimes dr + D(r, t)d\Omega. \quad (2.2)$$

Here the group orbits are labeled by  $t$  and  $r$ ,  $d\Omega$  is the standard metric on the unit 2-sphere, and  $A, B, C, D$  are arbitrary functions (aside from the conditions required to keep the metric non-degenerate and to give it the correct signature). We obtain the conditions for the invariant metric (2.2) to define a vacuum spacetime by requiring that this metric have vanishing Einstein tensor  $G_{ab}$ . This requirement yields a system of 10 PDEs in 2 independent variables for the four functions  $A, B, C, D$ :

$$G_{ab}[A, B, C, D] = 0, \quad (2.3)$$

but it turns out that only 4 of the equations are independent. Thus we obtain 4 reduced field equations for 4 reduced fields, from which one can obtain the Schwarzschild solution, *etc.* Now we can try our shortcut. Substitute the metric (2.2) into the Einstein-Hilbert

Lagrangian density,  $L = \sqrt{g}R$ , where  $R$  is the scalar curvature of the metric. Up to an irrelevant factor coming from the area element on the unit 2-sphere, the result is a Lagrangian density,  $\hat{L}$ , for the reduced fields. The Euler-Lagrange equations for the reduced fields obtained from  $\hat{L}$  are a system of 4 PDEs in 2 independent variables:

$$\frac{\delta \hat{S}}{\delta A} = \frac{\delta \hat{S}}{\delta B} = \frac{\delta \hat{S}}{\delta C} = \frac{\delta \hat{S}}{\delta D} = 0. \quad (2.4)$$

The equations (2.4) can be shown to be equivalent to the equations (2.3), so the reduced Lagrangian  $\hat{L}$  correctly describes the spherically symmetric vacuum spacetimes. Thus the symmetric criticality principle holds for spherically symmetric reductions of the vacuum Einstein equations.

The symmetric criticality principle for the spherical symmetry reduction of general relativity was used by Weyl to derive the Schwarzschild solution [6]. The principle is also being used in recent approaches to canonical quantization of midi-superspace models of spherically symmetric spacetimes (see *e.g.*, [7]).

As another example, let us consider homogeneous solutions to the vacuum Einstein equations. These mini-superspace cosmological models are obtained by restricting attention to spacetimes that admit a three-dimensional group  $G$  of isometries with orbits  $\Sigma$  that are leaves of a co-dimension 1 spacelike foliation of spacetime. We label the homogeneous hypersurfaces by  $t$ . The invariant metrics take the form

$$g = \alpha(t)dt \otimes dt + \beta_i(t)dt \otimes \omega^i + \gamma_{ij}(t)\omega^i \otimes \omega^j, \quad (2.5)$$

where  $\alpha$ ,  $\beta_i$ , and  $\gamma_{ij} = \gamma_{ji}$  are arbitrary functions of  $t$  (modulo non-degeneracy and signature of the metric), and  $\omega^i$ ,  $i = 1, 2, 3$ , are a basis of  $G$ -invariant 1-forms on  $\Sigma$ . The equations of motion for homogeneous vacuum metrics are a system of 10 ODEs for the 10 functions  $\alpha$ ,  $\beta_i$  and  $\gamma_{ij}$  obtained by demanding that the metric (2.5) have vanishing Einstein tensor:

$$G_{ab}[\alpha, \beta, \gamma] = 0. \quad (2.6)$$

Thus we obtain 10 reduced equations of motion for the 10 reduced fields. As before, we can try to obtain the reduced equations of motion from the reduced Lagrangian  $\hat{L}$ . Substituting

(2.5) into the Einstein-Hilbert Lagrangian and computing the Euler-Lagrange equations for the reduced fields  $\alpha$ ,  $\beta_i$  and  $\gamma_{ij}$  we again obtain 10 ODEs:

$$\frac{\delta \hat{S}}{\delta \alpha} = \frac{\delta \hat{S}}{\delta \beta_i} = \frac{\delta \hat{S}}{\delta \gamma_{ij}} = 0. \quad (2.7)$$

The equations (2.7) are equivalent to (2.6) only if the structure constants  $C_{ab}{}^c$  of  $G$  satisfy

$$C_{ab}{}^b = 0. \quad (2.8)$$

Homogeneity groups satisfying (2.8) have been given the picturesque name *class A*. (A more descriptive term for groups satisfying (2.8) is *unimodular* since such groups admit a bi-invariant volume form.) If  $G$  does not satisfy (2.8) it is, of course, called *class B*. We see that the class A models obey the symmetric criticality principle but that the class B models do not. Thus the Einstein-Hilbert variational principle for the Einstein equations fails to induce a variational principle for the class B homogeneous cosmological models. This annoying feature of homogeneous class B mini-superspaces has been known for a long time (see *e.g.*, [8] and [9] for a discussion). Needless to say, canonical quantization of mini-superspaces describing homogeneous cosmologies has been studied only for the class A models.

Experience with examples such as described above suggests that the validity of the reduced Lagrangian (or Hamiltonian) for a given mini- or midi-superspace has to be checked on a theory-by-theory and group-by-group basis. Fortunately, it is possible to give general conditions that are necessary and sufficient for the symmetric criticality principle to be valid for any field theory. Palais gives results along these lines in the context of  $G$ -invariant functions on Banach manifolds [4]. It is possible to give a somewhat more detailed set of conditions by specializing to local Lagrangian field theories [10]. Here I would like to give an informal statement of the main result from [10] on symmetric criticality for local gravitational field theories. To state this result we need the following data. The symmetry group being used for reduction is  $G$ . The orbits of  $G$  in spacetime  $M$  have dimension  $q$ . The isotropy (or stabilizer) group of a point  $x \in M$  is  $H_x \subset G$ . It is assumed that the isotropy

group of any given point is a subgroup of the Lorentz group since this is a necessary and sufficient condition for the (local) existence of a  $G$ -invariant metric [10]. The vector space of symmetric rank-2 tensors at a point  $x \in M$  is denoted by  $V_x$ . The vector space of  $H_x$ -invariant symmetric rank-2 tensors at a point  $x$  of spacetime is denoted by  $V_x^H$ . The annihilator of  $V_x^H$  (linear functions on  $V$  that vanish on  $V^H$ ) is denoted by  $(V_x^H)_0$ . Finally, the Lie algebra cohomology of  $G$  relative to  $H$  at degree  $q$  is denoted by  $\mathcal{H}^q(G, H)$ . This is the space of  $H$ -invariant closed *modulo* exact  $q$ -forms on the group manifold for  $G$ .

**Theorem.** *The principle of symmetric criticality is valid for any metric field theory derivable from a local Lagrangian density if and only if the following 2 conditions are satisfied at each point  $x$  in the region of spacetime under consideration.*

$$(1) \mathcal{H}^q(G, H_x) \neq 0.$$

$$(2) V_x^H \cap (V_x^H)_0 = 0.$$

The appearance of these 2 conditions can be understood from the 2 possible ways that symmetric criticality can fail. Recall that the first variation of an action functional is the sum of (i) a boundary term coming from an integration by parts, and (ii) a volume term in which the Euler-Lagrange expression occurs. The reduced Euler-Lagrange equations fail to define solutions of the original Euler-Lagrange equations if (a) the boundary term of the original variational expression fails to reduce to the boundary term for the reduced Lagrangian and/or (b) any (non-trivial) field equations coming from the volume term of the original Lagrangian disappear in the volume term for the reduced Lagrangian. If (a) occurs, then the reduced Euler-Lagrange equations pick up additional terms that render the Euler-Lagrange equations incorrect. If (b) occurs, then there will be non-trivial field equations that simply do not arise from the reduced Lagrangian. Condition (1) in the theorem, which is equivalent to requiring that the group orbits admit a bi- $G$ -invariant volume form, is necessary and sufficient to guarantee that (a) will not occur. Condition (2) is necessary and sufficient to prevent (b).



*Remarks:*

- When I gave this talk I left out the second condition (2) in the statement of the theorem. At the time, we thought that the existence of a  $G$ -invariant metric would prevent problem (b) from arising. Problem (b) *is* absent for field theories of a Riemannian metric, and condition (2) is not needed in that setting. But in general, and in particular for field theories of a Lorentzian metric, problem (b) can arise and condition (2) is needed.
- The conditions needed for the validity of the symmetric criticality principle may seem a little arcane, but they are in fact quite easy to check using elementary linear algebra and some manipulation of structure constants for the symmetry group.
- The theorem above gives conditions for validity of reduction of *any* Lagrangian. It is possible for the 2 conditions to fail for a given Lagrangian and still have symmetric criticality holding *for that particular Lagrangian*. The utility of the theorem is that it allows you to take a given symmetry reduction and check *a priori* whether one can correctly symmetry reduce at the level of the Lagrangian, irrespective of the choice of Lagrangian. Perhaps I should also point out that even if the symmetric criticality principle fails and the reduced Lagrangian does not correctly describe the reduced field equations, this does not mean that the reduced field equations do not admit a variational principle of some other type.
- It can be shown that the conditions of the theorem are satisfied if  $G$  is compact. In particular, it is always safe to reduce the Lagrangian via spherical symmetry (a fact often taken for granted in the physics literature!). If the group action is free, that is, has no (non-trivial) isotropy subgroups, then condition (2) in the theorem is trivially satisfied and condition (1) reduces to the statement that the Lie group is unimodular (2.8). Thus we recover the results of our two examples given above, and we see that these results are not specific to the Einstein field equations.
- While the theorem above is stated in the context of a local field theory of a metric, there is a straightforward generalization of the theorem to essentially any type of local field theory [4,10].

### III. TWO KILLING VECTOR MIDI-SUPERSPACES

Let us now focus on a particular class of midi-superspace models obtained by assuming the existence of 2 commuting Killing vector fields for the spacetimes of interest. These models have an infinite number of degrees of freedom and are equivalent to field theories in two-dimensions. Thus these are among the simplest of the midi-superspaces. We shall present the Hamiltonian formulation of these models and indicate that they are mathematically equivalent to parametrized field theories. Thus their quantization can be viewed as an instance of Dirac's problem of quantization on curved surfaces.

We will study two of the “2 Killing vector models”, namely, a cylindrically symmetric model and a toroidally symmetric model. The former is a generalization of the Einstein-Rosen wave model of Kuchař [3]. The latter is the “Gowdy model” [11]. We begin by defining the spacetime manifold and symmetry group.

#### *Cylindrical Symmetry*

Here the spacetime manifold is  $M = \mathbf{R} \times \mathbf{R}^3$  with cylindrical coordinates  $(t, x, \phi, z)$ , where

$$t \in (-\infty, \infty), \quad x \in (0, \infty), \quad \phi \in (0, 2\pi), \quad z \in (-\infty, \infty). \quad (3.1)$$

The symmetry group is generated by a translation, a rotation, and a discrete  $Z_2$ . The translation and rotation are generated by the vector fields  $(\frac{\partial}{\partial z}, \frac{\partial}{\partial \phi})$ ,

$$\phi \rightarrow \phi + \text{constant modulo } 2\pi, \quad z \rightarrow z + \text{constant}, \quad (3.2)$$

and the discrete transformation is

$$(t, x, \phi, z) \longrightarrow (t, x, 2\pi - \phi, -z). \quad (3.3)$$

#### *Toroidal Symmetry*

Here the spacetime manifold is  $M = \mathbf{R} \times \mathbf{T}^3$  with coordinates  $(t, x, y, z)$ , where

$$t \in (-\infty, \infty), \quad x, y, z \in (0, 2\pi). \quad (3.4)$$

Note that each of  $(x, y, z)$  are angular coordinates on a 3-torus. The symmetry group is generated by two rotations on the torus and a  $Z_2$  again. The rotations are generated by the vector fields  $(\frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ ,

$$y \rightarrow y + \text{constant modulo } 2\pi, \quad z \rightarrow z + \text{constant modulo } 2\pi, \quad (3.5)$$

and the discrete transformation is

$$(t, x, y, z) \longrightarrow (t, x, 2\pi - y, 2\pi - z). \quad (3.6)$$

In each case the discrete symmetry is designed to force the orthogonal distribution associated with the Killing vector fields to be integrable. In the usual terminology, we are considering spacetimes admitting two commuting Killing vector fields generating an “orthogonally transitive” group action.

It is straightforward to find the general form of the metrics admitting the toroidal and cylindrical symmetry groups. We present their line elements in the coordinates described above [12].

### *Cylindrical Symmetry*

$$ds^2 = \left[ -(N^\perp)^2 + e^{\gamma-\psi} (N^x)^2 \right] dt^2 + 2e^{\gamma-\psi} N^x dt dx + e^{\gamma-\psi} dx^2 + \Phi^2 e^{-\psi} d\phi^2 + e^\psi (dz + \tilde{\psi} d\phi)^2. \quad (3.7)$$

Each of the 6 fields entering into the components of the metric are functions of  $t$  and  $x$  only. We assume that  $\Phi > 0$ , that the spacetime gradient of  $\Phi$  is everywhere spacelike, and that  $N^\perp > 0$ . The reduced fields  $(N^\perp, N^x, \gamma, \Phi, \psi, \tilde{\psi})$  are otherwise unrestricted. The variables  $N^\perp$  and  $N^x$  are the lapse and shift for a symmetry compatible foliation (see [13] for a discussion of the 3+1 formalism). The Einstein equations are 6 non-linear PDEs for the six reduced fields. The “true degrees of freedom” of the model can be identified with the fields  $(\psi, \tilde{\psi})$  [12].

### *Toroidal Symmetry*

$$ds^2 = \left[ -(N^\perp)^2 + e^{\gamma-\psi} (N^x)^2 \right] dt^2 + 2e^{\gamma-\psi} N^x dt dx + e^{\gamma-\psi} dx^2 + \Phi^2 e^{-\psi} dy^2 + e^\psi (dz + \tilde{\psi} dy)^2. \quad (3.8)$$

Each of the 6 fields entering into the components of the metric are functions of  $t$  and  $x$  only. We assume that  $\Phi > 0$ , that the spacetime gradient of  $\Phi$  is everywhere timelike, and that  $N^\perp > 0$ . The reduced fields  $(N^\perp, N^x, \gamma, \Phi, \psi, \tilde{\psi})$  are otherwise unrestricted. The variables  $N^\perp$  and  $N^x$  are the lapse and shift for a symmetry compatible foliation. The Einstein equations are 6 non-linear PDEs for the six reduced fields. The “true degrees of freedom” of the model can be identified with the fields  $(\psi, \tilde{\psi})$  subject to a single “zero momentum” constraint and a single “point particle” degree of freedom [12].

Let us check that the 2 Killing vector midi-superspace models acquire a Hamiltonian structure from the Hamiltonian structure of the full theory (see [13,21,15,14] for a discussion of Hamiltonian gravity). This will follow if the symmetric criticality principle is valid for the cylindrical and toroidal symmetry reductions. We therefore check whether the two conditions from the theorem of the last section are satisfied for these symmetry reductions. That the first condition is satisfied follows from the fact that the group orbits admit an invariant volume form, *e.g.*,  $d\phi \wedge dz$  (cylindrical symmetry) or  $dy \wedge dz$  (toroidal symmetry). That the second condition is satisfied follows from the fact that the representation of the  $(Z_2)$  linear isotropy group at any point of the spacetime manifold is fully reducible. This second condition can also be checked by direct computation of the spaces  $V^H$  and  $V_0^H$ . We conclude that the symmetric criticality principle applies to cylindrical or toroidal symmetry reductions of any metric field theory. Therefore, it is permissible to derive the Hamiltonian formulation of these models by restricting the full Hamiltonian formulation to the chosen midi-superspace. This justifies the usual procedure, often found in the literature, in which the general form of the invariant metric is substituted into the ADM action to obtain the ADM action for the reduced theory.

It is possible to give a more or less unified treatment of the Hamiltonian formulation of

the models we are considering. The most succinct way to do this is to display the phase space (or “ADM”) action functional:

$$S = \int_{\hat{M}} \left( \pi_{\Phi} \dot{\Phi} + \pi_{\gamma} \dot{\gamma} + \pi_{\psi} \dot{\psi} + \pi_{\tilde{\psi}} \dot{\tilde{\psi}} - N^{\perp} \mathcal{H}_{\perp} - N^x \mathcal{H}_x \right) + \text{boundary term.} \quad (3.9)$$

The structure of (3.9) is as follows. The integral is over the space of orbits,  $\hat{M}$  (with coordinates  $(t, x)$ ), of the Killing vector fields. Thus  $\hat{M} = \mathbf{R} \times \mathbf{R}^+$  in the cylindrical symmetry case, and  $\hat{M} = \mathbf{R} \times S^1$  in the toroidal symmetry case. The critical points of this action functional define vacuum spacetimes with the prescribed symmetries. To find these critical points the action is to be varied with respect to the midi-superspace fields  $(\gamma, \Phi, \psi, \tilde{\psi})$ , their conjugate momenta  $(\pi_{\gamma}, \pi_{\Phi}, \pi_{\psi}, \pi_{\tilde{\psi}})$ , along with the lapse and shift  $(N^{\perp}, N^x)$ . The latter two variations lead to the (symmetry reduced) Hamiltonian and momentum constraints:

$$\mathcal{H}_{\perp} := e^{(\psi-\gamma)/2} \left[ -\pi_{\gamma} \pi_{\Phi} + 2\Phi'' - \Phi' \gamma' + \frac{1}{2}(\Phi \psi'^2 + \Phi^{-1} \pi_{\psi}^2) + \frac{1}{2}(\Phi e^{-2\psi} \pi_{\tilde{\psi}}^2 + \Phi^{-1} e^{2\psi} \tilde{\psi}'^2) \right] = 0, \quad (3.10)$$

$$\mathcal{H}_x := -2\pi'_{\gamma} + \pi_{\gamma} \gamma' + \pi_{\Phi} \Phi' + \pi_{\psi} \psi' + \pi_{\tilde{\psi}} \tilde{\psi}' = 0, \quad (3.11)$$

where a prime indicates a derivative with respect to the spatial coordinate  $x$ . The boundary term indicated in (3.9) only arises in the cylindrically symmetric case and is needed to render the action and Hamiltonian differentiable with appropriate boundary conditions. I refer you to [12] for details on this point.

The appearance of a pair of constraints on the phase space of the midi-superspace models reflects the presence of a gauge symmetry of the models with respect to a two-dimensional diffeomorphism group. This is the group of diffeomorphisms of the symmetry-reduced space-time manifold, *i.e.*, the space of orbits  $\hat{M}$  of the Killing vector fields. Indeed, one can check by direct computation that the Poisson algebra of the constraint functions  $(\mathcal{H}_{\perp}, \mathcal{H}_x)$  is the Dirac algebra of hypersurface (really, curve) deformations in a two-dimensional spacetime, which is the Hamiltonian expression of general covariance [15]. As in the full theory of

gravity, the Hamiltonian is (up to surface terms) built from the constraint functions, so that the constraint functions generate (almost all) of the dynamics. It is the existence of a Hamiltonian and momentum constraint that makes the 2 Killing vector midi-superspaces such excellent models of canonical quantum gravity.

Unfortunately, the constraint functions  $\mathcal{H}_\perp$  and  $\mathcal{H}_x$ , as they stand, are still rather intractable from the point of view of quantization. A straightforward approach *a la* Dirac [5] (see also [1] and [3]) would go as follows. Build the state space of the quantum theory as a space of functionals of the midi-superspace variables  $(\gamma, \Phi, \psi, \tilde{\psi})$  modulo spatial  $(x)$  diffeomorphisms. By taking the quotient with respect to the action of spatial diffeomorphisms we take into account the quantum form of the momentum constraint (3.11). What is left is the midi-superspace version of the Wheeler-DeWitt equation, in which one imposes the quantum form of the constraint (3.10), say, by trying to represent the midi-superspace variables as multiplication operators, representing their conjugate momenta as functional derivative operators, and then demanding that this quantization of  $\mathcal{H}_\perp$  annihilate physical states. No one seems to have made any progress using this most direct of approaches. One might say that the 2 Killing vector midi-superspaces model the situation in vacuum geometrodynamics all too well. To my knowledge, progress on these models has been made using 3 alternative approaches.

First, one can translate the midi-superspace model into the phase space description based upon Ashtekar variables [14]. As in the full theory of gravity, this leads to a significantly different set of strategies for the quantization of the constraints. This approach was initiated in [16] and some preliminary results obtained. Neville has developed this approach in some detail for the plane wave midi-superspaces [17]. I must direct you to these references for details.

Second, one can simply eliminate the diffeomorphism invariance (at the classical level) using coordinate conditions. In this approach, the constraints are solved classically and the issues of general covariance, constraint quantization, *etc.*, are eliminated from consideration. There is a nice gauge fixing for the models being considered here based upon the Einstein-

Rosen (cylindrical symmetry) or Gowdy (toroidal symmetry) coordinates. If the dynamics of the theory are restricted to foliations of spacetime adapted to these coordinate systems, then the dynamics of the theory are (modulo a few subtleties, see below) mathematically equivalent to that of a symmetry reduction of a wave map [18] from a flat three-dimensional spacetime to a two-dimensional Riemannian manifold of constant negative curvature. The wave map is provided by the fields  $\psi$  and  $\tilde{\psi}$ . In this interpretation of the symmetry reduced, gauge fixed theory, the space of orbits  $\hat{M}$  is viewed as a symmetry reduction of a flat three dimensional spacetime by a one dimensional group. By completely fixing the gauge, one can thus turn the quantization problem to the study of the quantum theory of wave maps on a flat spacetime. This point of view is the one taken in [19] and [20]. In particular, much progress has been made for the case where one assumes that the Killing vector fields are hypersurface orthogonal. This removes one “polarization” from the gravitational field and reduces the wave map to a single free scalar field, which is the Einstein-Rosen wave amplitude in the cylindrically symmetric case. Because the model has been reduced, in effect, to a free field theory, one can say quite a bit about the quantum theory. It is gratifying to be able to extract quantum information about spacetime geometry in this field-theoretic setting.

While the gauge-fixed quantum theory of the midi-superspace models has shed new light on possible physical properties of quantum geometry, the models fail to help us understand the full theory in one important respect. In the full theory one aspires to formulate the quantization in such a way as to preserve general covariance. This means one keeps the constraints in the theory and quantizes the constrained theory *a la* Dirac [5]. Presumably, the results obtained for the fully gauge fixed models arise as a specialization of the putative gauge invariant quantum theory. Without the ability to appeal to a gauge invariant formulation, it is not clear how to relate the results obtained via different gauge fixing methods. Indeed, it is hard to be sure that the gauge fixed theory has been quantized in a manner consistent with general covariance.

A third approach is possible, which still takes advantage of the wave map nature of the true degrees of freedom, but which preserves general covariance so as to provide a viable

model of Dirac constraint quantization of the full theory. This approach is a generalization of that used by Kuchař, and is based upon a canonical transformation that identifies the midi-superspace model with a *parametrized field theory* of wave maps. I would now like to describe this approach in more detail.

A parametrized field theory is a field theoretic generalization of a familiar construction from mechanics (see, e.g., [35]) in which one introduces a new, arbitrary time parameter  $\tau$  into the system and formally treats the true time  $t$  as a new degree of freedom which evolves in the new time parameter. Of course, time is not a degree of freedom: the time  $t$  is an arbitrary function of the parameter time,  $t = t(\tau)$ . This fact manifests itself in the appearance of a constraint in the Hamiltonian formulation that identifies the momentum conjugate to time with (minus) the canonical energy of the system. One can interpret the appearance of the constraint as reflecting a diffeomorphism gauge symmetry of the problem associated with the arbitrariness of the new time parameter  $\tau$ . The generalization of this formalism to field theory is reasonably straightforward [5,21,22]. In field theory, an instant of time is a Cauchy hypersurface. Given a field theory on a fixed spacetime background, one can express the theory in terms of an arbitrary foliation of the background and treat instants of time (Cauchy surfaces) as new dynamical variables. A Cauchy surface can be determined by giving its embedding in the given spacetime. For a four-dimensional spacetime, this means that one must specify 4 functions of 3 variables. These 4 functions, along with their canonical momenta, are added to the phase space of the field theory to obtain the parametrized field theory. As in the mechanical case, adding time to the phase space of the theory also adds constraints to the phase space. Four functions of the canonical variables must vanish. These constraints take the following form:

$$\mathcal{C}_\alpha(x) := P_\alpha(x) + h_\alpha[\phi, \pi, T](x) = 0. \quad (3.12)$$

The notation used in (3.12) is as follows. Coordinates on spacetime are denoted by  $T^\alpha$ . An embedding of a Cauchy surface  $\Sigma$  is given parametrically via

$$T^\alpha = T^\alpha(x), \quad (3.13)$$



where  $x^i$  are coordinates on  $\Sigma$ . The functions  $T^\alpha(x)$  are dynamical variables in the parametrized field theory. Their conjugate momenta are denoted by  $P_\alpha(x)$ . The true degrees of freedom of the theory are represented by the canonical variables  $(\phi, \pi)$ . The quantities  $h_\alpha[\phi, \pi, T](x)$  are the flux of energy-momentum at the spacetime point  $T^\alpha(x)$  associated with the Cauchy surface embedded by  $T^\alpha$ . In formulas:

$$h_\alpha = \sqrt{\gamma} n^\beta \Theta_{\alpha\beta}, \quad (3.14)$$

where  $\Theta_{\alpha\beta}$  is the energy momentum tensor for the fields  $(\phi, \pi)$ ,  $n^\alpha$  is the timelike unit normal to the hypersurface defined by  $T^\alpha(x)$  and  $\gamma_{ij}$  is the induced metric on that hypersurface.

As you can see, the constraint (3.12) identifies the variable conjugate to time (space) with minus the energy (momentum) density of the true degrees of freedom, in complete analogy with the mechanical version of the parametrized system. The presence of the constraints in the parametrized field theory reflect the existence of a four-dimensional diffeomorphism gauge symmetry for the theory. These constraints are “first class”; in fact the Poisson algebra of the constraints (3.12) is Abelian. This allows one to make a direct connection between the canonical transformations generated by the constraint functions and the action of spacetime diffeomorphisms on the parametrized field theory [22]. The Dirac algebra of hypersurface deformations is obtained by taking projections of the constraints (3.12) normally and tangentially to the Cauchy surfaces and computing their Poisson algebra.

To get a diffeomorphism symmetry for a field theory on a fixed background we have to add variables to the theory, that is, we have to “parametrize” the theory to make it generally covariant. Of course, in general relativity these variables are, in some sense, already there, and one often calls general relativity an “already parametrized field theory”. Unlike the case with an already parametrized theory such as general relativity, when parameterizing a field theory on a fixed background spacetime the constraints that appear have a very simple structure. They indicate explicitly that 4 canonical pairs are not truly dynamical. Moreover, the constraint functions generate the dynamical evolution of the true degrees of freedom as one deforms the embedding upon which the degrees of freedom are being mea-

sured. This leads to a natural approach to Dirac constraint quantization of a parametrized field theory. The embedding momenta are defined as variational derivative operators with respect to the embeddings  $X^\alpha(x)$ , and one must define the quantum energy-momentum flux as a self-adjoint operator on a Hilbert space of states for the true degrees of freedom. The quantum constraints then, at least formally, constitute a functional Schrödinger (or Tomonaga-Schwinger) equation

$$\left( \frac{1}{i} \frac{\delta}{\delta T^\alpha} + h_\alpha \right) |\Psi\rangle = 0, \quad (3.15)$$

which defines the dynamical evolution of the state vector  $|\Psi\rangle$  along an arbitrary foliation of spacetime.

We have remarked that by gauge fixing the midi-superspace models one ends up with a (symmetry reduction) of a field theory of wave maps on a flat spacetime. Such a field theory can be made generally covariant by the parametrization process sketched above, and it is natural to ask if this parametrized field theory has anything to do with the “already parametrized” midi-superspace model. The answer is affirmative. By generalizing Kuchař’s treatment of Einstein-Rosen waves, it is possible to identify the cylindrical and/or toroidal midi-superspace models with parametrized field theories of a one-dimensional symmetry reduction of wave maps from a flat three-dimensional spacetime to a two-dimensional Riemannian manifold of constant negative curvature. More precisely, given a slight modification of the phase spaces for the midi-superspace models being discussed (see the remark below), there exist canonical transformations identifying these models with parametrized field theories of wave maps [12]. Thus the constraints (3.10) and (3.11) of the toroidal or cylindrical midi-superspaces can be expressed in the parametrized field theory form (3.12), and a clear strategy for implementing these constraints in quantum theory is thus available. This strategy for quantization was explored by Kuchař, albeit at a rather formal level, for Einstein-Rosen waves in [3]. Teitelboim proposed that this same strategy could be used to formulate the canonical quantization of the full theory [23]. Limitations on this approach are discussed in [24].

To use these results to implement Dirac constraint quantization of the midi-superspaces being discussed here we must be able to construct a quantum field theory on a fixed (indeed, flat) spacetime that allows for dynamical evolution along arbitrary foliations of spacetime. Only relatively recently have results on the possibility of doing this become available. We will take up this issue in the next section.

*Remark:*

To make a rigorous identification of the midi-superspace models with a parametrized field theory, one must extend slightly the definition of the phase space for the midi-superspace models. The reason for this is that one is aspiring to use the intrinsic and extrinsic geometry of a hypersurface (the geometric interpretation of the gravitational phase space variables) to define how that hypersurface is embedded in spacetime. As it happens, the geometry of a hypersurface is not quite adequate to determine its embedding into spacetime. One must add a single variable to the phase space of the cylindrical symmetry model, and a pair of variables must be added to the toroidal symmetry model to allow for the identification with a parametrized field theory of wave maps. In the toroidal symmetry case there is also a single extra constraint that augments the usual constraints (3.12) of the parametrized field theory. The meaning of this constraint is that the total momentum for the wave maps must vanish. For details on all this, see [12].

#### IV. QUANTIZATION ON CURVED SURFACES

Let us summarize the discussion of the last section. Cylindrically symmetric and toroidally symmetric midi-superspace models of vacuum gravity are mathematically described by (a symmetry reduction of) parametrized field theories of wave maps on a flat spacetime. This allows us to turn the problem of Dirac constraint quantization of the midi-superspace models to that of finding a quantization of fields on a fixed background spacetime that allows for dynamical evolution along an arbitrary foliation of spacetime by Cauchy surfaces.

Remarkably, in the same 1964 monograph [5] in which Dirac described his methods for handling constrained Hamiltonian systems, he also considered the problem of canonically quantizing a field theory on Minkowski spacetime such that one could consistently evolve the state of the system from one arbitrary spacelike hypersurface to another. He called this the problem of “quantization on curved surfaces”. He formulated the problem in terms of the associated parametrized field theory and indicated that, in general, one could expect difficulties with consistency due to factor ordering problems in the quantum constraints (3.15).

As far as I can tell, it took over 20 years before an example of quantization on curved surfaces was worked out in any detail. I have in mind the work of Kuchař in [25], followed by work of Varadarajan and myself [26], which considers the problem in the context of a free scalar field theory on a flat two-dimensional background. The problem is already rather interesting (and has only been explored) for free fields, so let me try to describe what is going on in that case only.

Let us first consider classical time evolution for a free field  $\varphi$  on a globally hyperbolic spacetime  $(M, g)$ . The evolution is determined by the field equations which we write as

$$\Delta(\varphi) = 0, \tag{4.1}$$

where  $\Delta$  is a linear differential operator, *e.g.*,  $\varphi$  is a scalar field and  $\Delta = \nabla_a \nabla^a - m^2$  is the Klein-Gordon operator. Denote by  $\mathcal{S}$  the space of solutions to (4.1) with appropriate boundary conditions, say, compactly supported Cauchy data on any Cauchy surface  $\Sigma$ . Denote by  $\Gamma$  the space of Cauchy data for (4.1). Because the Cauchy problem is well-posed, there is for each Cauchy surface  $\Sigma$  an isomorphism  $e: \Gamma \rightarrow \mathcal{S}$  which takes Cauchy data on that surface and yields the unique solution to (4.1) with that data. The inverse,  $e^{-1}: \mathcal{S} \rightarrow \Gamma$ , takes a solution and yields its Cauchy data on  $\Sigma$ . Assuming that the free field has no gauge symmetries, there will exist a (weak) symplectic form on the space of solutions, that is, a skew-bilinear, non-degenerate map  $\Omega: \mathcal{S} \times \mathcal{S} \rightarrow \mathbf{R}$ . Given a Cauchy surface, the symplectic form on  $\mathcal{S}$  can be pulled back to  $\Gamma$  using  $e$  to define a symplectic form  $\omega: \Gamma \times \Gamma \rightarrow \mathbf{R}$ :

$$\omega = e^* \Omega. \quad (4.2)$$

It can be shown that  $\omega$  is independent of the choice of Cauchy surface used to define  $e$ . Either of the symplectic vector spaces  $(\mathcal{S}, \Omega)$  or  $(\Gamma, \omega)$  can be viewed as the phase space for the free field theory.

In order to discuss dynamics, we consider evolution from some initial time to some final time. Let us therefore fix an initial Cauchy hypersurface  $\Sigma_1$  and a final Cauchy hypersurface  $\Sigma_2$ . Associated with every such pair of instants of time there are maps  $e_1$  and  $e_2$  from  $\Gamma$  to  $\mathcal{S}$  and there is an isomorphism

$$\mathcal{T}_{12}: \mathcal{S} \rightarrow \mathcal{S} \quad (4.3)$$

given by

$$\mathcal{T}_{12} = e_1 \circ e_2^{-1}. \quad (4.4)$$

Given a solution to (4.1), *i.e.*, a point in  $\mathcal{S}$ , the linear map  $\mathcal{T}_{12}$  takes its Cauchy data on  $\Sigma_2$  and yields the solution that has this data on  $\Sigma_1$ . This mapping can be viewed as “time evolution” from  $\Sigma_1$  to  $\Sigma_2$  as represented on the space of solutions. To see what this time evolution means in terms of evolution of Cauchy data, let us use the isomorphism  $e_1$ , associated with the initial surface to identify  $\Gamma$  and  $\mathcal{S}$ . Using this identification, the mapping  $\mathcal{T}_{12}$  can be viewed as an isomorphism  $\tau_{12}: \Gamma \rightarrow \Gamma$  given by

$$\tau_{12} := e_1^{-1} \circ \mathcal{T}_{12} \circ e_1 = e_2^{-1} \circ e_1. \quad (4.5)$$

The map  $\tau_{12}$  defines dynamical evolution on  $\Gamma$  by taking data on the initial surface, evolving it into a solution of (4.1), and then yielding the new data on the final surface.

Whether viewed as a map on  $\mathcal{S}$  or on  $\Gamma$ , time evolution preserves the respective symplectic structure:

$$\mathcal{T}_{12}^* \Omega = \Omega, \quad \tau_{12}^* \omega = \omega. \quad (4.6)$$

This is, of course, just the familiar result for Hamiltonian systems that “time evolution is a canonical transformation”. The problem of quantization on curved surfaces can be viewed as

the problem of transporting the classical dynamical structure just described to the quantum description of the field theory.

Wald gives a very general prescription for constructing a Fock space quantization of a linear field in a globally hyperbolic spacetime [27]. The key step is the construction of the one-particle Hilbert space, from which the Fock space  $\mathcal{F}$  is constructed in the usual way. The possible choices of one-particle Hilbert space correspond to choices of a (suitable) inner product on the symplectic space  $(\mathcal{S}, \Omega)$  (or  $(\Gamma, \omega)$ ). Given an inner product on  $\mathcal{S}$ , one obtains the Fock space  $\mathcal{F}$  and (densely defined) field operators  $\Phi(\varphi)$ , labeled by elements of  $\mathcal{S}$ , satisfying the CCR algebra

$$[\Phi(\varphi_1), \Phi(\varphi_2)] = -i\Omega(\varphi_1, \varphi_2). \quad (4.7)$$

Dynamical evolution from  $\Sigma_1$  to  $\Sigma_2$  in the Heisenberg picture corresponds to the algebra automorphism

$$\mathcal{T}_{12} \cdot \Phi(\phi) := \Phi(\mathcal{T}_{12} \cdot \phi). \quad (4.8)$$

Normally, one expects that dynamical evolution is implemented by a unitary transformation  $U_{12}: \mathcal{F} \rightarrow \mathcal{F}$ , such that

$$U_{12}^{-1} \Phi U_{12} = \mathcal{T}_{12} \cdot \Phi. \quad (4.9)$$

Assuming this is the case (but see below), given a state  $|\psi\rangle$  prepared at the time defined by  $\Sigma_1$  we can define the state at time  $\Sigma_2$  (in the Schrodinger picture) as  $U_{12}|\psi\rangle$ . The Tomonaga-Schwinger equation (3.15), at least formally, describes the change in  $U_{12}|\psi\rangle$  as the surface  $\Sigma_2$  is deformed in spacetime. One can therefore use  $U_{12}$  to define the “physical states” in the Dirac constraint quantization of parametrized field theory. In detail, fix once and for all an initial surface  $\Sigma_0$ . Choose a state  $|\psi_0\rangle$  in the Fock space and apply the unitary transformation corresponding to evolution from  $\Sigma_0$  to an arbitrary final surface  $\Sigma$ . The result can be viewed as a  $\Sigma$ -dependent state satisfying (formally) the constraints (3.15). This “physical state” is determined by its value  $|\psi_0\rangle$  on the initial surface, and all physical

states arise by varying the choice of  $|\psi_0\rangle$ . Of course, what I have described is just a “many-fingered time” generalization of the standard approach to solving the Schrödinger equation. Insofar as parametrized field theories accurately model already parametrized theories such as general relativity, we can interpret the Wheeler-DeWitt equation as a Schrödinger equation in disguise and try to find the physical quantum states using the approach just described.

Unfortunately, unitary implementability of the symplectic transformation  $\mathcal{T}_{12}$  is not guaranteed. It is well-known that not all canonical transformations can be unitarily implemented in quantum mechanics. In quantum field theory the presence of an infinite number of degrees of freedom can even prevent unitary implementability of linear canonical transformations, such as we are considering here [29]. Failure of unitary implementability of time evolution is not unheard of in the context of quantum field theory in a non-stationary curved spacetime [27], where it is usually associated with infinite particle production by the gravitational field. However, there the whole issue is complicated by the fact that the fields are interacting with a prescribed gravitational field and, in general, there is no preferred quantization of the classical theory. A much simpler situation one might consider is that of free fields in flat spacetime; there is then a timelike Killing vector field that defines a preferred inner product on  $\mathcal{S}$  and a consequent preferred quantization. Of course this is precisely the setting one finds oneself in when considering the simplest of the midi-superspace models described in the previous section. Following the lead of Kuchař [25], in [26] M. Varadarajan and I consider unitary implementability of dynamical evolution along arbitrary foliations, as well as the definition and solution of the quantum constraints (3.15), for a free scalar field propagating on a flat two-dimensional spacetime. Let me now summarize some of the salient results.

The quantum theory for a free, massless scalar field on a flat, cylindrical spacetime is defined in the standard way. Unitary implementability of the transformation  $\mathcal{T}_{12}$  can be verified for  $\Sigma_1$  and  $\Sigma_2$  being any Cauchy surfaces (actually, Cauchy circles). It is possible to construct the unitary transformation explicitly and check the status of the putative quantum constraint (3.15). Let  $T: S^1 \rightarrow \mathbf{R} \times S^1$  be an embedding of a Cauchy surface. We consider the image,  $|\Psi(T)\rangle$ , of the unitary map taking any initial state on any initial Cauchy surface

to the surface embedded by  $T$ . We find that this state vector satisfies a quantum constraint of the form [26]

$$\left( \frac{1}{i} \frac{\delta}{\delta T^\alpha} + h_\alpha + A_\alpha[T] \right) |\Psi(T)\rangle = 0. \quad (4.10)$$

Here  $h_\alpha$  is the normal-ordered (with respect to the usual vacuum) energy-momentum current density in the Schrodinger picture. The term  $A_\alpha$  is a multiple of the identity operator which depends on the embedding  $T$ ; thus this term is a “time-dependent  $c$ -number”. Based upon general arguments that take into account the Schwinger terms in the algebra of energy-momentum tensor components, Kuchař proposed that the quantum constraints for this model should have such a term, and our direct computation verifies this proposal. Thus there is a “quantum correction” to the classical constraints, which is an interesting phenomenon to encounter in such a simple model.

Other, related models one can consider are obtained by adding a mass to the scalar field and/or changing the topology of the spacetime to  $\mathbf{R}^2$ . Adding a mass does not alter the unitary implementability of the “many-fingered time” dynamical evolution. Allowing the Cauchy surfaces to be non-compact requires asymptotically (extrinsically) flat Cauchy surfaces to be used in order to guarantee unitary evolution in the massive case. Massless fields on  $\mathbf{R}^2$  lead to the usual infrared difficulties, and haven’t been explored as yet. Presumably, these results can be generalized to any free fields in two spacetime dimensions. Aside from the technical difficulty with massless fields on  $\mathbf{R}^2$ , it appears that the problem of quantization on curved surfaces is satisfactorily solved for free fields in two spacetime dimensions.

It is now tempting to suppose that these results have a straightforward generalization to free field theories on higher-dimensional flat spacetimes. However, in quantum field theory it seems that nothing should be taken for granted. If  $\Sigma_1$  and/or  $\Sigma_2$  are suitably generic, the symplectic transformation representing dynamical evolution will *not* be unitarily implementable. A sketch of this result is given in [26]; details will appear in the near future [28]. The situation is rather like the Van Hove obstruction to unitary implementability of the group of canonical transformations [30]. In quantum mechanics it is well known that



only a subset of the canonical transformations can be represented as unitary transformations on a Hilbert space. For a free field theory on flat spacetime one can unitarily implement dynamical evolution between Cauchy surfaces related by an action of the Poincaré group, but more general transformations are not supported by the Fock space. The problem seems to be that the group of hypersurface deformations for a free field does not have a unitary representation on the standard free field Fock space except for two-dimensional spacetimes. For two-dimensional field theories, the many-fingered time evolution can be viewed as an action of the conformal group on spacetime [31]. The conformal group for a two-dimensional spacetime is, in turn, built from a pair of one-dimensional diffeomorphism groups, which have unitary representations on Fock space [32], at least for free fields. In higher-dimensions this simple picture is simply not available. Thus Dirac's problem of quantization on curved surfaces remains an open problem for free fields on a flat spacetime of dimension greater than two. By this I mean that it is not known how to find a Hilbert space representation of the CCR that allows for unitary dynamical evolution along arbitrary foliations and which reduces to the usual dynamical evolution when using foliations by flat hypersurfaces. It is worth noting that there seems to be no obstruction to quantization on curved surfaces using the more general approach to quantum theory provided by the apparatus of algebraic quantum theory (see [33,27] for a description of this approach to quantum field theory).

At this point it is appropriate to recall the motivation for our discussion of quantization on curved surfaces, that is, quantization of parametrized field theories. We were able, classically, to view the two Killing vector midi-superspaces as the parameterization of field theories on a flat three-dimensional spacetime. It is then natural to try to define the "Wheeler-DeWitt equation" as the functional Schrödinger equation (3.15). We saw that quantization of parametrized fields in dimensions three and higher is problematic; what is to be done about the quantization of these midi-superspace models *a la* Dirac? This question is made more interesting by the fact that the reduced field theories for these models, while naturally viewed as symmetry reductions of three-dimensional theories, are in effect two-dimensional theories. Is it possible that these models are just two-dimensional enough

in their behavior to allow the quantization via parametrized field theory? Recent work of M. Varadarajan suggests that this is the case [34]. Work is in progress on this issue, and I hope that soon we will know whether the quantization of midi-superspace models of canonical quantum gravity proposed in [3] and [12] is possible. If so, results such as [19,20] can be viewed as arising from a generally covariant canonical quantization of the gravitational field, and these models can shed light on the nature of quantum geometry in the framework of Dirac's quantization of constrained systems.

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